Invariant Hilbert Scheme Resolution of Popov's $SL(2)$-varieties

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The Invariant Hilbert Scheme

Question (general case).

Does the **Hilbert–Chow morphism**

$$\gamma : \mathrm{Hilb}_h^G(X) \longrightarrow X/\!\!/G \quad (\clubsuit)$$

give a **desingularization** of $X/\!\!/G$?

Definition (Alexeev–Brion).
The **invariant Hilbert scheme** is defined as follows:

$$\mathrm{Hilb}_h^G(X) = \left\{ Z \subset X \;\middle|\; \begin{array}{l} Z \text{ is a closed } G\text{-subscheme of } X; \\ \mathbb{C}[Z] \cong \bigoplus_{M \in \mathrm{Irr}(G)} M^{\oplus h(M)} \text{ as a } G\text{-module} \end{array} \right\}.$$

- G: a reductive algebraic group;
- X: an affine G-variety;
- $h : \mathrm{Irr}(G) \to \mathbb{N}$: a **Hilbert function**.

Remark.
$\mathrm{Hilb}_h^G(X)$ is a generalization of G-$\mathrm{Hilb}(X)$ for a finite group G.

- $\pi : X \longrightarrow X/\!\!/G$: the quotient morphism.
- $h :=$ Hilbert function of the **flat locus** $W \longrightarrow W/\!\!/G$ of π.

$$\rightsquigarrow \begin{cases} \bullet\ \mathcal{H}^{main} := \overline{\gamma^{-1}(W)}\text{: the \textbf{main component} of } \mathrm{Hilb}_h^G(X). \\ \bullet\ \gamma|_{\mathcal{H}^{main}} : \mathcal{H}^{main} \longrightarrow X/\!\!/G = \mathrm{Spec}\,\mathbb{C}[X]^G\text{: \textbf{proj. birat.}} \end{cases}$$

Popov's $SL(2)$-varieties

Popov gave a complete classification of 3-dim. affine normal quasihomog. $SL(2)$-varieties.

Theorem (Popov).
There is a one to one correspondence:

$$\left\{ \begin{array}{c} \text{3-dim. affine normal quasihomog.} \\ SL(2)\text{-var. with a fixed point} \end{array} \right\} \longleftrightarrow \{\mathbb{Q} \cap (0,1)\} \times \mathbb{N}$$

$$E_{l,m} \qquad \longleftrightarrow \qquad (l,m)$$

$E_{l,m}$ contains three $SL(2)$-orbits: $E_{l,m} = U \cup D \cup \{O\}$.

- U: the **dense open orbit**;
- D: a 2-dim. orbit;
- O: a unique $SL(2)$-inv. **singular point**.

Popov's $SL(2)$-varieties as a GIT Quotient

Batyrev and Haddad proved that Popov's variety has a description as an affine quotient.

Theorem (Batyrev–Haddad).

- $E_{l,m}$: 3-dim. affine normal quasihomog. $SL(2)$-var.;
- $\mathbb{C}^5 \supset H_{q-p} := (X_0^{q-p} = X_1X_4 - X_2X_3)$.

Then,

$$E_{l,m} \cong H_{q-p}/\!\!/(\mathbb{C}^* \times \mu_m).$$

Remark. Actions of $SL(2)$, $\mathbb{C}^*$, and μ_m on $\mathbb{C}^5$ are given as follows:

- $SL(2) \ni \forall g$,
$$\begin{pmatrix} a & c \\ b & d \end{pmatrix} \cdot \left(X_0, \begin{pmatrix} X_1 & X_3 \\ X_2 & X_4 \end{pmatrix} \right) := \left(X_0, \begin{pmatrix} aX_1 + cX_2 & aX_3 + cX_4 \\ bX_1 + dX_2 & bX_3 + dX_4 \end{pmatrix} \right);$$
- $\mathbb{C}^* \ni \forall t, \quad t \cdot (X_0,\ X_1,\ X_2,\ X_3,\ X_4) := (tX_0,\ t^{-p}X_1,\ t^{-p}X_2,\ t^qX_3,\ t^qX_4)$;
- $\mu_m \ni \forall \xi, \quad \xi \cdot (X_0,\ X_1,\ X_2,\ X_3,\ X_4) := (X_0,\ \xi^{-1}X_1,\ \xi^{-1}X_2,\ \xi X_3,\ \xi X_4)$.

Question (our case). Apply ($\clubsuit$) with

$$X = H_{q-p}, \ G = \mathbb{C}^* \times \mu_m:$$

$$\gamma : \mathrm{Hilb}_h^{\mathbb{C}^* \times \mu_m}(H_{q-p}) \longrightarrow E_{l,m}$$

Spherical Geometry

We use the **spherical geometry** of $E_{l,m}$ and $Bl_O^\omega(E_{l,m})$ to study

$$\gamma : \mathrm{Hilb}_h^{\mathbb{C}^* \times \mu_m}(H_{q-p}) \longrightarrow E_{l,m}.$$

Theorem (Batyrev–Haddad).
$E_{l,m}$ and $Bl_O^\omega(E_{l,m})$ are **spherical** $SL(2) \times \mathbb{C}^*$-varieties w.r.t. $B \times \mathbb{C}^*$.

- [Batyrev–Haddad] also computed the **colored fan** of $E_{l,m}$, $Bl_O^\omega(E_{l,m})$.

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- [Brion–Pauer] Local structure th. for **toroidal spherical varieties**.

$\Downarrow$

Theorem (Batyrev–Haddad).
(i) $\exists!\ C \cong \mathbb{P}^1$: a closed $SL(2)$-orbit of $Bl_O^\omega(E_{l,m})$.
(ii) Along C, $Bl_O^\omega(E_{l,m})$ is locally isomorphic to $\mathbb{C} \times \mathbb{C}^2/\mu_b$.
(iii) $b = 1 \iff E_{l,m}$: **toric**.

Main Results

The Hilbert–Chow morphism decomposes as follows:

$$\begin{tikzcd} \mathcal{H}^{main} \arrow[r, "\exists\,\psi"] \arrow[dr, "\gamma|_{\mathcal{H}^{main}}"'] & Bl_O^\omega(E_{l,m}) \arrow[d] \\ & E_{l,m} \end{tikzcd}$$

Main Theorem

(i) $E_{l,m}$: **toric**

$$\Rightarrow \begin{cases} \gamma : \mathcal{H}^{main} \longrightarrow E_{l,m} \text{ is a \textbf{resol.}} \\ \gamma^{-1}(O) \cong \mathbb{P}^1 \times \mathbb{P}^1. \end{cases}$$

(ii) $\psi : \mathcal{H}^{main} \cong Bl_O^\omega(E_{l,m})$
$\iff E_{l,m}$: **toric.**

Outline of the Proof

Step 1. Determine the generators of ideals in $\gamma^{-1}(U)$.

Step 2. Construct ψ (use **Step 1.** + irred. decomp. of $\mathbb{C}[H_{q-p}]$):

$$\mathcal{H}^{main} \xrightarrow{\psi} E_{l,m} \times \mathbb{P}^1 \times \mathbb{P}^1.$$

Step 3. Use the spherical geometry of $E_{l,m}$ and $Bl_O^\omega(E_{l,m})$ to show

$$\psi(\mathcal{H}^{main}) \cong Bl_O^\omega(E_{l,m}) \hookrightarrow E_{l,m} \times \mathbb{P}^1 \times \mathbb{P}^1.$$

Step 4. $E_{l,m}$: toric $\Rightarrow \psi$ is a cl. imm. $[\because]$ Description of generators for $\forall I_Z \in \mathcal{H}^{main}$.

Step 5. $E_{l,m}$: non-toric $\Rightarrow \mathcal{H}^{main} \ncong Bl_O^\omega(E_{l,m})$ $[\because]$ Calculation of flat limits of ideals.

Work in Progress

[Main Theorem] $E_{l,m}$: **non-toric** $\Rightarrow \mathcal{H}^{main} \ncong Bl_O^\omega(E_{l,m})$.
$\rightsquigarrow$ We need a **further blow-up**.

- [Batyrev–Haddad] $E_{l,m}$: **non-toric** $\Rightarrow Bl_O^\omega(E_{l,m})$ is **sing.** along C.
- $\mathbb{C}^2/\mu_b$ has a **minimal resol.** described by **Hirzebruch–Jung continued fraction**.

Conjecture

$E_{l,m}$: **non-toric**

$\Rightarrow \mathcal{H}^{main} \longrightarrow Bl_O^\omega(E_{l,m})$ is a **minimal resol.**